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On the Size of Polynomials with Curved Majorant

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Let $C_n(\varphi)$ denote all polynomials of degree n majorized by a positive C^2 function φ on $[-1, 1]$, $n = 0, 1, 2, \ldots$. We establish that for every $r \in (0, 1)$, there is an integer $N(r, \varphi) > 0$, such that, for all $n \geqslant N(r, \varphi)$, the polynomials in $C_n(\varphi)$ could be as large as φ on $[-r, r]$, i.e.,

$$\max_{p_n \in C_n(\varphi)} |p_n(x)| = \varphi(x),$$

for all $x \in [-r, r]$ and $n \geqslant N(r, \varphi)$. This is related to a result of Newman and Rivlin [6].

1. INTRODUCTION

Let $\varphi(x) > 0$ on $(-1, 1)$. Define

$$C_n(\varphi) := \{p_n \in \mathscr{P}_n \mid |p_n(x)| \leqslant \varphi(x),\ -1 < x < 1\},$$

where $\mathscr{P}_n$ denotes the set of polynomials of degree at most n. Following Rahman (cf. [8]), such $p_n \in C_n(\varphi)$ is called a polynomial with curved majorant φ. Turán raised the question about the size of $|p_n'(x)|$ for p_n in $C_n(\varphi)$ when $\varphi(x) = (1 - x^2)^{1/2}$. Estimation has been made (cf. [7–9]), but the "precise value" is still not known for all x. In this article, instead of the size of $|p_n'(x)|$, we are concerned with the size of $|p_n(x)|$ when $p_n \in C_n(\varphi)$. Naturally,

$$|p_n(x)| \leqslant \varphi(x), \qquad -1 < x < 1.$$

The question is whether the equality is always possible and if not, at what points the equality holds. This is closely related to the weighted approximation.

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Our results are inspired by a paper of Newman and Rivlin [6], in which they answered the above question when $\varphi(x) = (1-x^2)^{\pm 1/2}$.

This paper is organized as follows. The statements of our results are given in Section 2. Their proofs appear in Section 3.

2. Statements of Results

Our first result shows that on subintervals $[-r, r]$ $(0 < r < 1)$, some polynomials with curved majorant φ can be as large as φ provided that their degrees are high enough $(n \geqslant N(r, \varphi))$. More precisely, we have the following.

Proposition 1. *Assume $\varphi > 0$ and φ'' is continuous in $(-1, 1)$. Then for every $r \in (0, 1)$, there exists an integer $N = N(r, \varphi) > 0$ depending on r and φ only, such that*

$$\max_{p \in C_n(\varphi)} |p(x)| = \varphi(x), \tag{1}$$

for all $x \in [-r, r]$ and $n \geqslant N$.

Remark 1. From a result of Ivanov and Totik [1, Example 1], one can see that for certain function φ, Eq. (1) may fail for small n.

Remark 2. When $\lim_{|x| \to 1^-} \varphi(x) = +\infty$, we must have $\lim_{r \to 1^-} N = +\infty$. We prove this fact at the end of Section 3.

To state our next result, we need some preparations. Assume function $\varphi\colon [-1, 1] \to (0, +\infty]$ is continuous (including the situations in which either $\lim_{x \to 1^-} \varphi(x) = +\infty$, or $\lim_{x \to -1^+} \varphi(x) = +\infty$, or both). Then it is well known (cf. [5]) that there exists (smallest) interval $[s(n), t(n)] \subseteq [-1, 1]$ such that

$$\|p\|_\varphi := \sup_{x \in (-1, 1)} \left|\frac{p(x)}{\varphi(x)}\right| = \max_{x \in [s(n), t(n)]} \left|\frac{p(x)}{\varphi(x)}\right|$$

for all $p \in \mathscr{P}_n$, with

$$t(n) < 1 \qquad \text{iff} \quad \lim_{x \to 1^-} \varphi(x) = +\infty,$$

and

$$s(n) > -1 \qquad \text{iff} \quad \lim_{x \to -1^+} \varphi(x) = +\infty.$$

Note that $\{x^k/\varphi(x)\}_{k=0}^n$ is a Chebychev system over $[s(n), t(n)]$. Thus, there exists (uniquely) the so-called weighted Chebychev polynomial of degree n (with respect to φ), $T_{n,\varphi}(x) = x^n + \cdots \in \mathscr{P}_n$, characterized by

$$\|T_{n,\varphi}\|_\varphi = \inf_{p(x) = x^n + \cdots \in \mathscr{P}_n} \|p\|_\varphi.$$

By Chebychev's maximal equioscillation theorem, there are points ξ_k, $k = 1, 2, ..., n+1$, such that

$$s(n) \leqslant \xi_{n+1} < \xi_n < \cdots < \xi_1 \leqslant t(n)$$

and

$$T_{n,\varphi}(\xi_k)/\varphi(\xi_k) = (-1)^{k+1} \|T_{n,\varphi}\|_\varphi, \qquad k = 1, 2, ..., n+1.$$

A set of points like $\{\xi_k\}_{k=1}^{n+1}$ will be called a set of points of equioscillation of $T_{n,\varphi}(x)/\varphi(x)$. Generally, such a set of points of equioscillation is not unique. Denote

$$\hat{\xi}_{n+1}(n) = \sup \xi_{n+1} \qquad \text{and} \qquad \tilde{\xi}_1(n) = \inf \xi_1,$$

where the "sup" and "inf" are taken over all sets of points of equioscillation with ξ_{n+1} and ξ_1 being the smallest and largest points, respectively. We write $\hat{\xi}_{n+1}(\tilde{\xi}_1)$ for $\hat{\xi}_{n+1}(n)$ (resp. $\tilde{\xi}_1(n)$) when there is no confusion. A compactness argument yields that there are $\{\hat{\xi}_k\}_{k=1}^n$ and $\{\tilde{\xi}_k\}_{k=2}^{n+1}$ such that both $\{\hat{\xi}_k\}_{k=1}^{n+1}$ and $\{\tilde{\xi}_k\}_{k=1}^{n+1}$ form sets of points of equioscillation of $T_{n,\varphi}(x)/\varphi(x)$. So the "sup" and "inf" in the definitions of $\hat{\xi}_{n+1}$ and $\tilde{\xi}_1$ can be replaced by "max" and "min", respectively.

Define $\hat{T}_{n,\varphi}(x) := T_{n,\varphi}(x)/\|T_{n,\varphi}\|_\varphi$, then $|\hat{T}_{n,\varphi}(x)| \leqslant \varphi(x)$, $-1 < x < 1$; so $\hat{T}_{n,\varphi} \in C_n(\varphi)$. Our next result says that $\hat{T}_{n,\varphi}$ has the largest absolute value outside interval $(\hat{\xi}_{n+1}, \tilde{\xi}_1)$ among all polynomials in $C_n(\varphi)$.

Proposition 2. *Assume that function* $\varphi\colon [-1, 1] \to (0, +\infty]$ *is continuous. Then for* $x \in (-\infty, \hat{\xi}_{n+1}] \cup [\tilde{\xi}_1, +\infty)$,

$$\max_{p \in C_n(\varphi)} |p(x)| = |\hat{T}_{n,\varphi}(x)|. \tag{2}$$

Remark 3. When $\varphi(x) = (1 + x^2)^{\pm 1/2}$, Eq. (2) was proved by Newman and Rivlin in [6]. When $\varphi(x) = (1-x)^{-\alpha}(1+x)^{-\beta}$ with $\alpha, \beta \geqslant 0$, it was established by Lachance *et al.* in [2]. Both proofs in these two articles employed the Lagrange's interpolation formula. It turns out that the same approach works for the general case with proper modifications. For completeness, we give a sketch of the proof of Proposition 2 in the next section.

As a consequence of Propositions 1 and 2, we can say something about the smallest and the largest extreme points of the weighted Chebychev polynomial $T_{n,\varphi}(x)/\varphi(x)$.

PROPOSITION 3. *Assume function* $\varphi\colon [-1, 1] \to (0, +\infty]$ *is continuous, and assume* φ'' *is continuous on* $(-1, 1)$. *Then*

$$\lim_{n\to\infty} \tilde{\xi}_1(n) = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \hat{\xi}_{n+1}(n) = -1.$$

Remark 4. This proposition should be compared with a result of Lubinsky and Saff (cf. [4, p. 58, Corollary 8.2]). There, in the case of $\varphi(x) = e^{Q(x)}$ defined on the whole real line with φ'' continuous, under the additional condition that φ' is positive in $(0, \infty)$, the order of the largest extreme point of the weighted Chebychev polynomial was estimated.

We now state the main result, which is the consequence of the combination of the above three technical propositions.

THEOREM. *Assume function* $\varphi\colon [-1, 1] \to (0, +\infty]$ *is continuous, and assume* φ'' *is continuous on* $(-1, 1)$. *Let* $r \in (0, 1)$ *be given. Then there exists an integer* $N' = N'(r, \varphi) > 0$ *such that, for* $b \geqslant N'$, *we have*

$$\hat{\xi}_{n+1} < -r, \qquad \tilde{\xi}_1 > r$$

and

$$\max_{p\in C_n(\varphi)} |p(x)| = \begin{cases} \varphi(x), & \text{if } x \in [-r, r], \\ |\hat{T}_{n,\varphi}(x)|, & \text{if } x \in (-\infty, \hat{\xi}_{n+1}] \cup [\tilde{\xi}_1, +\infty). \end{cases}$$

The above theorem makes one wonder what happens to

$$\max_{p\in C_n(\varphi)} |p(x)|$$

for $x \in (\tilde{\xi}_{n+1}, -r) \cup (r, \tilde{\xi}_1)$ $(n \geqslant N')$. More generally, the following question is of interest.

Question. For a given integer $n \geqslant 0$, what is the subset of $[-1, 1]$ of the points at which $\max_{p\in C_n(\varphi)} |p(x)| = \varphi(x)$? And on the remaining part of $[-1, 1]$, what are the values $\max_{p\in C_n}(\varphi)\, |p(x)|$ and the polynomials yielding the maximum values?

For a special family of functions φ, this question has been solved in [3]. The general case is unsolved.

3. Proofs

Our proof of Proposition 1 involves, for every $a \in [-r, r]$, the construction of a polynomial $p_a \in C_n(\varphi)$ (when n is large enough) such that

$$p_a(a) = \varphi(a).$$

The classical Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$, which is the fastest increasing polynomial outside $[-1, 1]$, plays a very important role in our construction. The following lemma lists some of the properties of T_n, which are used in our proof.

Lemma 1. *For $|x| > 1$, we have the following statements.*

(i) $T_n(x) = \frac{1}{2}[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n]$.

(ii) $T_n'(x) = (n/2\sqrt{x^2 - 1})[(x + \sqrt{x^2 - 1})^n - (x - \sqrt{x^2 - 1})^n]$.

(iii) $T_n''(x) = (n/2(x^2 - 1)^{3/2})[(x + \sqrt{x^2 - 1})^n (n\sqrt{x^2 - 1} - x) + (x - \sqrt{x^2 - 1})^n (n\sqrt{x^2 - 1} + x)]$.

(iv) *Both* $n^2 T_n(x)/T_n''(x)$ *and* $nT_n'(x)/T_n''(x)$ *converge locally uniformly in* $(-\infty, -1) \cup (1, +\infty)$.

Proof. (i) is well-known, and it actually holds for all $x \in (-\infty, +\infty)$ for a suitable choice of the branch of the square root. (ii)–(iv) are obtained by straightforward computations. ∎

Let $r \in (0, 1)$ be given. For $\delta \in (0, 1 - r)$, define

$$f(x) := \frac{4 + \delta^2 - 2x^2}{4 - \delta^2}$$

and

$$q_n(x) := T_{[n/2]}(f(x)) = T_{[n/2]}\left(\frac{4 + \delta^2 - 2x^2}{4 - \delta^2}\right).$$

Then f maps $[-1, 1]$ onto $[(2 + \delta^2)/(4 - \delta^2), (4 + \delta^2)/(4 - \delta^2)]$, $\max_{\delta \leqslant |x| \leqslant 1} |q_n(x)| = 1$, and $\min_{|x| \leqslant \delta/\sqrt{2}} q_n(x) = T_{[n/2]}(4/(4 - \delta^2)) \to +\infty$ $(n \to +\infty)$.

Note that, using Lemma 1,

$$\lim_{n \to +\infty} \frac{q_n''(0)}{nq_n(0)} = \lim_{n \to +\infty} \frac{-T'_{[n/2]}((4 + \delta^2)/(4 - \delta^2))\, 4/(4 - \delta^2)}{nT_{[n/2]}((4 + \delta^2)/(4 - \delta^2))} = -\frac{1}{2\delta} < 0,$$

and

$$q_n''\left(\frac{\delta}{\sqrt{2}}\right) = T_{[n/2]}''\left(\frac{4}{4-\delta^2}\right)\frac{8\delta^2}{(4-\delta^2)^2} - T_{[n/2]}'\left(\frac{4}{4-\delta^2}\right)\frac{4}{4-\delta^2} > 0,$$

for n large enough. If we denote

$$m_0 := \min_{x \in [-r-\delta, r+\delta]} \varphi(x) \qquad \text{and} \qquad m_2 := \min\{0, \min_{x \in [-r-\delta, r+\delta]} \varphi''(x)\},$$

then $m_0 > 0$ and $m_2 \leqslant 0$, and we can find $N_1 = N_1(r, \delta, \varphi)$ such that

$$\frac{q_n''(0)}{q_n(0)} < \frac{m_2}{m_0} \qquad \text{and} \qquad \frac{q_n''(\delta/\sqrt{2})}{q_n(0)} > 0,$$

for every $n \geqslant N_1$.

Now, for every $n \geqslant N_1$, choose $x_n \in (0, \delta/\sqrt{2})$ such that

$$x_n := \min\left\{x \in \left(0, \frac{\delta}{\sqrt{2}}\right) : \frac{q_n''(x)}{q_n(0)} = \frac{m_2}{m_0}\right\}.$$

Then $q_n''(x) < q_n''(x_n) \leqslant 0$ for $|x| < x_n$ and $n \geqslant N_1$.

Our next lemma reveals some asymptotic properties of x_n.

LEMMA 2. *We have the following limit relations*:

$$\lim_{n \to +\infty} \sqrt{n}\, x_n = \sqrt{2\delta}, \tag{3}$$

$$\lim_{n \to +\infty} \frac{q_n(x_n)}{q_n(0)} = \frac{1}{\sqrt{e}}, \tag{4}$$

and

$$\lim_{n \to +\infty} \frac{1}{\sqrt{n}} \frac{q_n'(x_n)}{q_n(0)} = -\frac{1}{\sqrt{2\delta e}}. \tag{5}$$

Proof. We first show

$$\lim_{n \to +\infty} x_n = 0. \tag{6}$$

Assume $\varepsilon \in [0, \delta/\sqrt{2}]$ is a limit point of $\{x_n\}_{n \geqslant N_1} \subseteq [0, \delta/\sqrt{2}]$; then there exists $\Lambda \subseteq \{n\}_{n \geqslant N_1}$ such that

$$\lim_{\substack{n \to +\infty \\ n \in \Lambda}} x_n = \varepsilon.$$

From the definition of x_n, we can write

$$\left(\frac{4x_n}{4-\delta^2}\right)^2 = \frac{T'_{[n/2]}(f(x_n))}{T''_{[n/2]}(f(x_n))}\frac{4}{4-\delta^2} + \frac{m_2}{m_0}\frac{T_{[n/2]}(f(0))}{T''_{[n/2]}(f(x_n))}. \tag{7}$$

Note that $m_2 \leqslant 0$, $m_0 > 0$, $T_{[n/2]}(f(0)) > 0$ and $T''_{[n/2]}(f(x_n)) > 0$, for n large enough. So the second term on the right side of (7) is non-positive; thus

$$\left(\frac{4x_n}{4-\delta^2}\right)^2 \leqslant \frac{T'_{[n/2]}(f(x_n))}{T''_{[n/2]}(f(x_n))}\frac{4}{4-\delta^2}, \tag{8}$$

for n large enough. From Lemma 1, it can be verified that

$$\lim_{\substack{n \to +\infty \\ n \in \Lambda}} \frac{n}{2}\frac{T'_{[n/2]}(f(x_n))}{T''_{[n/2]}(f(x_n))} = \frac{2\sqrt{(4-\varepsilon^2)(\delta^2-\varepsilon^2)}}{4-\delta^2} > 0. \tag{9}$$

Thus, by letting $n \to +\infty$ and $n \in \Lambda$ in (8), we get

$$\left(\frac{4\varepsilon}{4-\delta^2}\right)^2 \leqslant 0,$$

or

$$\frac{4\varepsilon}{4-\delta^2} = 0.$$

So $\varepsilon = 0$. Consequently, Eq. (6) holds. Using this fact in (9), we obtain

$$\lim_{n \to +\infty} \frac{n}{2}\frac{T'_{[n/2]}(f(x_n))}{T''_{[n/2]}(f(x_n))} = \frac{4\delta}{4-\delta^2}. \tag{10}$$

Now, multiplying (8) by n and letting $n \to +\infty$ yield $\limsup_{n \to +\infty} nx_n^2 \leqslant 2\delta$. Together with Lemma 1, this implies

$$\limsup_{n \to +\infty} \left(\frac{n}{2}\right)^2 \frac{T_{[n/2]}(f(0))}{T''_{[n/2]}(f(x_n))} \leqslant \frac{16^2\sqrt{e}}{(4-\delta^2)^2}.$$

Therefore, $\limsup_{n \to +\infty} nT_{[n/2]}(f(0))/T''_{[n/2]}(f(x_n)) \leqslant 0$. But $T_{[n/2]}(f(0))/T''_{[n/2]}(f(x_n)) > 0$, so

$$\lim_{n \to +\infty} n\frac{T_{[n/2]}(f(0))}{T''_{[n/2]}(f(x_n))} = 0. \tag{11}$$

When we multiply n on both sides of (7) and let $n \to +\infty$, using (10) and (11), we can get

$$\lim_{n \to +\infty} n \left(\frac{4x_n}{4-\delta^2} \right)^2 = \frac{32\delta}{(4-\delta^2)^2},$$

which is equivalent to (3).

For the proof of (4), write

$$\begin{aligned} \frac{q_n(x_n)}{q_n(0)} &= \frac{T_{[n/2]}((4+\delta^2-2x_n^2)/(4-\delta^2))}{T_{[n/2]}((4+\delta^2)/(4-\delta^2))} \\ &= \frac{\left[\begin{array}{l} ((\sqrt{4-x_n^2}+\sqrt{\delta^2-x_n^2})^2/(4-\delta^2))^{[n/2]} + ((4-\delta^2)/(\sqrt{4-x_n^2} \\ \quad + \sqrt{\delta^2-x_n^2})^2)^{[n/2]} \end{array} \right]}{((2+\delta)/(2-\delta))^{[n/2]} + ((2-\delta)/(2+\delta))^{[n/2]}}. \end{aligned}$$

So

$$\lim_{n \to +\infty} \frac{q_n(x_n)}{q_n(0)} = \frac{1}{\lim_{n \to +\infty} ((2+\delta)/(\sqrt{4-x_n^2}+\sqrt{\delta^2-x_n^2}))^{2[n/2]}}.$$

Using (3), we find

$$\lim_{n \to +\infty} \left(\frac{2+\delta}{\sqrt{4-x_n^2}+\sqrt{\delta^2-x_n^2}} \right)^{2[n/2]} = \sqrt{e}; \tag{12}$$

thus

$$\lim_{n \to +\infty} \frac{q_n(x_n)}{q_n(0)} = \frac{1}{\sqrt{e}},$$

which is (4).

The proof of (5) is very similar to that of (4). We have

$$\begin{aligned} \lim_{n \to +\infty} \frac{1}{\sqrt{n}} \frac{q_n'(x_n)}{q_n(0)} &= -\frac{1}{\sqrt{2\delta}} \frac{1}{\lim_{n \to +\infty} ((2+\delta)/(\sqrt{4-x_n^2}+\sqrt{\delta^2-x_n^2})^{2[n/2]}} \\ &= -\frac{1}{\sqrt{2\delta e}}, \end{aligned}$$

by (12). ∎

Proof of Proposition 1. In addition to all the notations introduced above, we also need the following quantities:

$$M := \max_{x \in [-r-\delta, r+\delta]} \varphi(x) \quad \text{and} \quad M' := \max_{x \in [-r-\delta, r+\delta]} |\varphi'(x)|.$$

Recall that $m_0 :=_{x \in [-r-\delta, r+\delta]} \varphi(x)$. Since $\lim_{\delta \to 0^+} m_0 = \min_{|x| \leqslant r} \varphi(x) > 0$, we can choose $\delta = \delta(r, \varphi) \in (0, 1-r)$ so small that

$$\frac{m_0}{\delta \sqrt{e}} > 2M'. \tag{13}$$

Then we choose $N_2 = N_2(r, \varphi) > 0$ such that

$$m_0 \frac{|q_n'(x_n)|}{q_n(0)} > M', \tag{14}$$

$$\frac{m_0}{q_n(0)} \frac{|1 - q_n(x_n)|}{\delta - x_n} > M', \tag{15}$$

and

$$\frac{M}{q_n(0)} + M'x_n \leqslant m_0, \tag{16}$$

for every $n \geqslant N_2$. The existence of such N_2 is guaranteed by (5), (4) and (13), and (3). From (4) and (5), we can again find $N_3 = N_3(r, \varphi) > 0$ so large that

$$|q_n'(x_n)| \geqslant \frac{|1 - q_n(x_n)|}{\delta - x_n}, \tag{17}$$

for every $n \geqslant N_3$.

We now show that $N = N(r, \varphi) := \max(N_1, N_2, N_3)$ is the N described in Proposition 1.

For $n \geqslant N$, define

$$p_n(x) := \frac{Mq_n(x)}{q_n(0)}.$$

For each $a \in [-r, r]$, define

$$\lambda = \lambda(a) := \frac{\varphi(a)}{M},$$

then $\lambda \in [m_0/M, 1]$. Now choose $b \in [a - x_n, a + x_n]$ such that

$$\lambda p_n'(b - a) = \varphi'(a).$$

Such number b exists because $\lambda p_n'(x)$ is an odd function and

$$|\lambda p_n'(x_n)| \geqslant \frac{m_0}{M} \frac{M\,|q_n'(x_n)|}{q_n(0)} > M',$$

by (14). If $\varphi'(a) \leqslant 0$, then $b \geqslant a$; and if $\varphi'(a) > 0$, then $b < a$. We only need to consider the case when $\varphi'(a) \leqslant 0$; the case when $\varphi'(a) > 0$ can be deduced either directly by using similar argument or from transformation $x \to -x$.

Construct $p_a(x)$ as follows:

$$p_a(x) := \lambda[p_n(x+b-2a) + M - p_n(b-a)].$$

We have

$$p_a(a) = \lambda M = \varphi(a)$$

and

$$p'_a(a) = \lambda p'_n(b-a) = \varphi'(a).$$

We claim that

$$p_a \in C_n(\varphi). \tag{18}$$

We divide the proof of claim (18) into three cases according to the location of $x \in (-1, 1)$.

Case 1. $|x+b-2a| \leqslant x_n$.

In this case, we have $x \in [-r-\delta, r+\delta]$, $|p_a(x)| = p_a(x)$, and for some ξ satisfying $|\xi + b - 2a| \leqslant x_n$,

$$\begin{aligned}
|p_a(x)| &= p_a(x) = \varphi(x) + p_a(x) - \varphi(x) \\
&= \varphi(x) + \frac{1}{2}(p''_a(\xi) - \varphi''(\xi))(x-a)^2 \\
&\leqslant \varphi(x) + \frac{1}{2}\left(\frac{\lambda M m_2}{m_0} - m_2\right)(x-a)^2 \\
&\leqslant \varphi(x) + \frac{1}{2}\left(\frac{m_0 M m_2}{M m_0} - m_2\right)(x-a)^2 = \varphi(x),
\end{aligned}$$

where we have used the facts that $m_2 \leqslant 0$ and

$$p''_a(\xi) = \frac{\lambda M q''_n(\xi + b - 2a)}{q_n(0)} \leqslant \frac{\lambda M q''_n(x_n)}{q_n(0)} = \frac{\lambda M m_2}{m_0}.$$

Case 2. $x_n < |x+b-2a| \leqslant \delta$.

In this case, we still have $|p_a(x)| = p_a(x)$. Assume to the contrary of (18) that for some x' satisfying $x_n - b + 2a < x' \leqslant \delta - b + 2a$,

$$p_a(x') \geqslant \varphi(x').$$

Then there is $\zeta \in (x_n - b + 2a, x')$ such that

$$\varphi'(\zeta) = \frac{\varphi(x') - \varphi(x_n - b + 2a)}{x' - x_n + b - 2a} \leqslant \frac{p_a(x') - p_a(x_n - b + 2a)}{x' - x_n + b - 2a}.$$

Since (17) implies that the graph of p_a over the interval $[x_n - b + 2a, \delta - b + 2a]$ is always below the segment connecting points $(x_n - b + 2a, p_a(x_n - b + 2a))$ and $(\delta - b + 2a, p_a(\delta - b + 2a))$, we see that

$$\frac{p_a(x') - p_a(x_n - b + 2a)}{x' - x_n + b - 2a} \leqslant \frac{p_a(\delta - b + 2a) - p_a(x_n - b + 2a)}{\delta - x_n}.$$

Hence

$$\begin{aligned}\varphi'(\zeta) &\leqslant \frac{p_a(\delta - b + 2a) - p_a(x_n - b + 2a)}{\delta - x_n} = \frac{\lambda M(1 - q_n(x_n))}{q_n(0)(\delta - x_n)} \\ &\leqslant \frac{m_0(1 - q_n(x_n))}{q_n(0)(\delta - x_n)} < -M',\end{aligned}$$

according to (15). But the definition of M' gives $|\varphi'(\zeta)| \leqslant M'$, so we get a contradiction. Similarly, one can show that there is no x'' satisfying $-\delta - b + 2a \leqslant x'' < -x_n - b + 2a$ such that

$$p_a(x'') \geqslant \varphi(x'').$$

So, in Case 2, we always have

$$0 \leqslant p_a(x) < \varphi(x).$$

Case 3. $\delta - b + 2a \leqslant x < 1$ or $-1 < x \leqslant -\delta - b + 2a$.

We have $|x + b - 2a| \geqslant \delta$ and

$$\begin{aligned}|p_a(x)| &= \lambda \left| \frac{Mq_n(x + b - 2a)}{q_n(0)} + M - \frac{Mq_n(b - a)}{q_n(0)} \right| \\ &= \frac{\lambda M}{q_n(0)} |q_n(x + b - 2a) - q_n'(\xi)(b - a)| \qquad (\xi \in (0, b - a)) \\ &\leqslant \frac{\lambda M}{q_n(0)} + \frac{\lambda M}{q_n(0)} |q_n'(b - a)| \, x_n \qquad (|q_n'(\xi)| \leqslant |q_n'(b - a)|) \\ &= \frac{\varphi(a)}{q_n(0)} + |\varphi'(a)| \, x_n \leqslant \frac{M}{q_n(0)} + M'x_n \leqslant m_0.\end{aligned}$$

Here in the last step we have used (16). So $|p_a(x)| \leqslant m_0 \leqslant \varphi(x)$ in this case.

The above three cases cover all points x in $(-1, 1)$; hence we have $|p_a(x)| \leqslant \varphi(x)$ for all $x \in (-1, 1)$, which is equivalent to claim (18). This completes our proof of Proposition 1. ∎

Proof of Proposition 2. Let $-1 \leqslant \xi_{n+1} < \xi_n < \cdots < \xi_1 \leqslant 1$ be a set of points of equioscillation of $T_{n,\varphi}(x)/\varphi(x)$. Define $w(x) := \prod_{k=1}^{n+1}(x-\xi_k)$. Then, for $p \in C_n(\varphi)$, Lagrange's interpolation formula yields

$$p(x) = \sum_{k=1}^{n+1} \frac{w(x)\,p(\xi_k)}{w'(\xi_k)(x-\xi_k)}.$$

So

$$|p(x)| \leqslant |w(x)| \sum_{k=1}^{n+1} \frac{|p(\xi_k)|}{|w'(\xi_k)(x-\xi_k)|} \leqslant |w(x)| \sum_{k=1}^{n+1} \frac{\varphi(\xi_k)}{|w'(\xi_k)(x-\xi_k)|}. \quad (19)$$

But $|w'(\xi_k)| = (-1)^{k+1} w'(\xi_k)$ and $\hat{T}_{n,\varphi}(\xi_k) = (-1)^{k+1}\varphi(\xi_k)$, for $k = 1, 2, ..., n+1$. Thus, the right side of (19) equals $|\hat{T}_{n,\varphi}(x)|$ when $x \notin [\xi_{n+1}, \xi_1]$. Now substituting $\{\xi_k\}_{k=1}^{n+1}$ by $\{\hat{\xi}_k\}_{k=1}^{n+1}$ and $\{\tilde{\xi}_k\}_{k=1}^{n+1}$, we obtain (2) for $x \notin [\hat{\xi}_{n+1}, \tilde{\xi}_1]$. ∎

Proof of Proposition 3. If there is an infinite set $\Lambda \subseteq \{n\}_{n\geqslant 1}$ such that

$$\tilde{\xi} < r_0 < 1, \qquad n \in \Lambda, \quad (20)$$

then for $n \in \Lambda$ and $n \geqslant N(r_0, \varphi)$, with $N(r_0, \varphi)$ as defined in Proposition 1,

$$\max_{p \in C_n(\varphi)} |p(x)| = \varphi(x), \qquad x \in [-r_0, r_0],$$

by Proposition 1. But Proposition 2 gives

$$\max_{p \in C_n(\varphi)} |p(x)| = \hat{T}_{n,\varphi}(x), \qquad x \geqslant \tilde{\xi}_1(n).$$

It then follows from (20) that

$$\varphi(x) = \hat{T}_{n,\varphi}(x), \qquad x \in [\tilde{\xi}_1(n), r_0],$$

for all n with $n \in \Lambda$ and $n \geqslant N(r_0, \varphi)$. So

$$\hat{T}_{n,\varphi}(x) = \hat{T}_{m,\varphi}(x), \qquad x \in [\max\{\tilde{\xi}_1(n), \tilde{\xi}_1(m)\}, r_0],$$

for all $m, n \in \Lambda$ and $m, n \geqslant N(r_0, \varphi)$. Thus $\hat{T}_{n,\varphi}(x) \equiv \hat{T}_{m,\varphi}(x)$, for all $m, n \in \Lambda$ and $m, n \geqslant N(r_0, \varphi)$, which is impossible when $m \neq n$. ∎

Proof of Remark 2. Without loss of generality, we can assume $N(r, \varphi)$ is non-decreasing in r. If $\lim_{|x| \to 1^-} \varphi(x) = +\infty$, and $\lim_{r \to 1^-} N(r, \varphi) \neq +\infty$, then, on the one hand, we must have

$$|\hat{T}_{n, \varphi}(x)| < \varphi(x), \tag{21}$$

for $|x|$ close enough to 1; on the other hand, there is a positive K such that

$$N(r, \varphi) \leqslant K, \tag{22}$$

for all $r \in (0, 1)$.

From (21), we see that $\tilde{\xi}_1(n) < 1$ and $\hat{\xi}_{n+1}(n) > -1$, and thus, by Proposition 2,

$$\max_{p \in C_n(\varphi)} |p(x)| = |\hat{T}_{n, \varphi}(x)|,$$

for $r(n) := \max\{|\tilde{\xi}_1(n), |\hat{\xi}_{n+1}(n)|\} \leqslant |x|$. But (22) and Proposition 1 would imply

$$\max_{p \in C_n(\varphi)} |p(x)| = \varphi(x), \qquad x \in (-1, 1),$$

for every $n \geqslant K$. Hence

$$|\hat{T}_{n, \varphi}(x)| = \varphi(x), \qquad r(n) \leqslant |x| < 1,$$

for every $n \geqslant K$, which implies particularly

$$\hat{T}_{K, \varphi}(x) = \hat{T}_{K+1, \varphi}(x), \qquad \max\{r(K), r(K+1)\} \leqslant x < 1,$$

which is impossible. ▮

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